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Complex dynamics in a ODE model related to phase transition

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Abstract Motivated by some recent studies on the Allen–Cahn phase transition model with a periodic nonautonomous term, we prove the existence of complex dynamics for the second order equation

$$-\ddot{x} + (1 + \varepsilon^{-1} A(t))G'(x) = 0,$$

where $A(t)$ is a nonnegative T-periodic function and $\varepsilon > 0$ is sufficiently small. More precisely, we find a full symbolic dynamics made by solutions which oscillate between *any* two different strict local minima x_0 and x_1 of $G(x)$. Such solutions stay close to x_0 or x_1 in some fixed intervals, according to any prescribed coin tossing sequence. For convenience in the exposition we consider (without loss of generality) the case $x_0 = 0$ and $x_1 = 1$.

Keywords periodic solutions · non-autonomous equations · Allen-Cahn equation · complex dynamics

Mathematics Subject Classification (2000) 34C25 · 34C28 · 54H20

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1 Introduction

Let $G : [0,1] \to \mathbb{R}$ be a differentiable function with locally Lipschitz continuous derivative $g(x) := G'(x)$ satisfying the following condition

$(G1)$ $G'(0) = 0 = G'(1)$ and there exist a_0, b_0 with $0 < a_0 < b_0 < 1$ such that

$$g(x) > 0 \quad \forall\, x \in \,]0, a_0] \quad \text{and} \quad g(x) < 0 \quad \forall\, x \in [b_0, 1[\,.$$

Let $A : \mathbb{R} \to \mathbb{R}$ be a T-periodic locally integrable function such that for some $\tau \in \,]0, T[$ it holds that

$(A1)$ $A(t) = 0$ for a. e. $t \in [\tau, T]$ and $A(t) > 0$ for a. e. $t \in [0, \tau]$.

In this paper, we study the second order nonlinear scalar ODE

$$\ddot{x} - w_\varepsilon(t) G'(x) = 0, \tag{1}$$

where, for $\varepsilon > 0$,

$$w_\varepsilon(t) := 1 + \frac{A(t)}{\varepsilon}\,.$$

Solutions of (1) are meant in the Carathéodory setting.

The study of these equations is motivated by the search of stationary solutions to some parabolic PDEs which are used in physical models of phase transition. A classical example is given by the Allen-Cahn equation introduced in [4]. In such models a typical potential $G(x)$ is a double well function as in the real Ginzburg-Landau equation. The presence of nonconstant weight functions accounts for models describing heterogeneous materials. In recent years a great deal of interests has been devoted to the study of multiple solutions satisfying different boundary conditions (see, for instance [1], [2], [3], [5], [6], [9], [17], [18] and the references therein).

Our interest for equation (1) is motivated by recent works by Byeon and Rabinowitz [6], [7], [8], [9] concerning the equation

$$-\Delta u + A_\varepsilon G'(u) = 0, \quad x \in \mathbb{R}^N, \tag{2}$$

where G is a double well potential of the form $G(u) = u^2(1-u)^2$ and

$$A_\varepsilon(x) := 1 + \frac{A(x)}{\varepsilon},$$

where $A(x)$ is a nontrivial non-negative function which is 1-periodic with respect to $x_1, \ldots, x_N$ and such that the support of $A|_{[0,1]^N}$ is contained in $]0,1[^N$. It was shown in [6] that there is an infinitude of mixed states that shadow 0 and 1 in any prescribed way on a spatially periodic array of sets (from the Introduction in [8]). Further improvements of this result were obtained in [9], by producing several other solutions of mountain pass type.

In the present work we consider a simpler situation with respect to the case of (2), in fact we deal with the one-dimensional case $N = 1$. On the other hand, we obtain analogous results with a completely different approach which rely on the theory of topological horseshoes [11] applied to planar dynamical systems. Our main results (Theorem 1 and Theorem 2) require a minimal set of assumptions on the potential. In particular, in our first result, we only suppose the existence of two strict local minima for the potential which are conventionally indicated as 0 and 1. Such local minima are neither required to be consecutive ones, nor at the same energy level. Indeed, we have:

Theorem 1 *Assume* $(G1)$ *and* $(A1)$*. For every pair* (a,b) *with* $0 < a \leq a_0$ *and* $b_0 \leq b < 1$*, there exists* $\varepsilon^* = \varepsilon(a,b) > 0$ *such that, for each fixed* $\varepsilon \in \,]0,\varepsilon^*[$ *the following property holds.*
For each nontrivial two-sided sequence $\boldsymbol{s} = (s_n)_{n\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$*, there exists at least one solution* $u(t) = u_{\boldsymbol{s},\varepsilon}(t)$ *of* (1) *with the following properties:*

1. $0 < u(t) < 1$ *for all* $t \in \mathbb{R}$*;*
2. *for all* $n \in \mathbb{Z}$ *one has that*

$$\begin{cases} 0 < u(t) \leq a & \text{if } s_n = 0 \\ b \leq u(t) < 1 & \text{if } s_n = 1 \end{cases} \qquad \forall t \in [nT, nT+\tau];$$

3. u *is* kT*-periodic if the sequence* $\boldsymbol{s}$ *is* k*-periodic for some* $k \in \mathbb{N}$*.*

In particular, as $\varepsilon \to 0^+$*, we have that:*

$$u_{\boldsymbol{s},\varepsilon} \to s_n \quad \text{uniformly on } [nT, nT+\tau] \text{ for each } n \in \mathbb{Z}.$$

In the trivial cases $\boldsymbol{s} = (0)_{n\in\mathbb{Z}}$ and $\boldsymbol{s} = (1)_{n\in\mathbb{Z}}$ we can only provide the trivial solutions $u \equiv 0$ and $u \equiv 1$, respectively.

Our result can be applied to any two strict local minima of the potential $G(x)$ in equation (1) without any other assumption on G. In particular, we do not assume that $G(0) = G(1)$ (a condition which sometimes has been required in related papers). Indeed, if G is a potential with several (possibly infinitely many) strict local minima, we can take any pair $\{x_0, x_1\}$ with $x_0 < x_1$ of such local minima and obtain a complex dynamics in the interval $]x_0, x_1[$ of the form described in Theorem 1. Another feature of our result is that we could allow the weight function $A(t)$ to vanish at some points in $]0,\tau[$, provided that there is no subinterval of $]0,\tau[$, where $A(\cdot)$ vanishes identically. However, our method can be easily adapted also to deal with the case in which the shape of the function $A(t)$ is made by a finite number of positive humps separated by some intervals where $A(\cdot)$ vanishes identically. This is briefly described at the end of Section 3. Finally, we point out that the constant ε^* can be estimated in terms of the coefficients of the equation (see the determination of ε_0 in Lemma 5). Even if we find useful for our computations to exploit some properties of the conservative equation $x'' + G'(x) = 0$ (for example, in using the energy level lines as comparison trajectories), however our method of proof is of topological nature and do not rely on the Hamiltonian/variational structure of (1). As a consequence, conclusions 1, 2, 3 of Theorem 1 are still true for an equation of the form

$$x'' + cx' + w_\varepsilon(t)G'(x) = 0$$

provided that c is a sufficiently small constant (depending on ε).

Using a classical approach, the study of (1) will be performed by means of the analysis of the equivalent system in the phase plane:

$$\begin{cases} \dot{x} = y \\ \dot{y} = w_\varepsilon(t) g(x). \end{cases} \tag{3}$$

For such a system, we will show that the associated Poincaré map has a rich dynamics.

The present paper is organized as follows. In Section 2 we recall the main topological tools which are used in the proof of our theorems. Namely, we give a brief survey of the so-called *stretching along the paths* (SAP) method introduced in [13] and further developed in a series of articles [12], [14], [15], [16]. Section 3 is devoted to the proof of Theorem 1 and to some of its immediate extensions. Subsequently, in Section 4, we propose a refinement of the results obtained in our main theorem from the point of view of the oscillatory properties of the solutions. This is obtained in Theorem 2, by imposing some extra assumptions on the potential $G(x)$ and also on the weight function $A(t)$.

Throughout the article, the following basic set of notation is used: We denote by $\mathbb{Z}$ and $\mathbb{N}$ the sets of integers and nonnegative integers, respectively.

2 Topological tools

In this section, we briefly recall some topological results concerning the method of *stretching along the paths* (SAP). The general theory has been developed for domains which are homeomorphic images of cylindrical sets in a Banach space (see [16]), however, for the purpose of the present paper, we just expose some basic facts in the simplified setting of planar maps.

By an *oriented rectangle* we mean a pair $\widetilde{\mathcal{R}} := (\mathcal{R}, \mathcal{R}^-)$, where $\mathcal{R} \subset \mathbb{R}^2$ is a homeomorphic image of the unit square $[0,1]^2$ and $\mathcal{R}^- \subset \partial\mathcal{R}$ is the union of two disjoint compact arcs denoted by $\mathcal{R}^{\text{left}}$ and $\mathcal{R}^{\text{right}}$. Consider now a continuous map $\Psi : \mathcal{D}_\Psi (\subset \mathbb{R}^2) \to \mathbb{R}^2$. Given two oriented rectangles $\widetilde{\mathcal{A}} := (\mathcal{A}, \mathcal{A}^-)$, $\widetilde{\mathcal{B}} := (\mathcal{B}, \mathcal{B}^-)$, and a compact subset $\mathcal{K}$ of $\mathcal{A} \cap \mathcal{D}_\Psi$, we say that the pair $(\mathcal{K}, \Psi)$ *stretches* $\widetilde{\mathcal{A}}$ *to* $\widetilde{\mathcal{B}}$ *along the paths* and write $(\mathcal{K}, \Psi) : \widetilde{\mathcal{A}} \Bumpeq\!\!\!\!\longrightarrow \widetilde{\mathcal{B}}$ if for each continuous curve $\gamma : [0,1] \to \mathcal{A}$ with $\gamma(0) \in \mathcal{A}^{\text{left}}$ and $\gamma(1) \in \mathcal{A}^{\text{right}}$ there exist $t', t'' \in [0,1]$ (with $t' < t''$) such that

1. $\gamma(t) \in \mathcal{K}$ and $\Psi(\gamma(t)) \in \mathcal{B}$ for all $t \in [t', t'']$;
2. $\Psi(\gamma(t'))$ and $\Psi(\gamma(t''))$ belong to different components of $\mathcal{B}^-$.

Usually the curve γ is called a path and its restriction to $[t', t'']$ a sub-path. We also say that Ψ *stretches* $\widetilde{\mathcal{A}}$ *to* $\widetilde{\mathcal{B}}$ *along the paths with crossing number* m and write $\Psi : \widetilde{\mathcal{A}} \Bumpeq\!\!\!\!\longrightarrow^m \widetilde{\mathcal{B}}$, if there exist $m \geq 2$ pairwise disjoint compact subsets $\mathcal{K}_1 \ldots, \mathcal{K}_m$ of $\mathcal{A} \cap \mathcal{D}_\Psi$, such that $(\mathcal{K}_i, \Psi) : \widetilde{\mathcal{A}} \Bumpeq\!\!\!\!\longrightarrow \widetilde{\mathcal{B}}$ for each $i = 1, \ldots, m$.

The SAP technique allows to prove the existence of fixed points for Ψ in the set $\mathcal{K}$, when $(\mathcal{K}, \Psi) : \widetilde{\mathcal{R}} \Bumpeq\!\!\!\!\longrightarrow \widetilde{\mathcal{R}}$, and, moreover, to detect the presence of a full symbolic dynamics on m symbols when $\Psi : \widetilde{\mathcal{R}} \Bumpeq\!\!\!\!\!\!\!\longrightarrow^m \widetilde{\mathcal{R}}$, for some $m \geq 2$. It can be interpreted in the context of the theory of so-called topological horseshoes, a name that is usually given to those theories that propose to extend the prototypical geometric scheme of Smale's horseshoe in a topological setting.

We present below some results which will be then applied to the Poincaré map associated to (1). For the sake of completeness in the exposition, we also introduce the set $\Sigma_m := \{0, \dots, m-1\}^{\mathbb{Z}}$ of two-sided sequences of $m \geq 2$ symbols with its standard metric and the (Bernoulli) shift automorphism $\sigma : \Sigma_m \to \Sigma_m$ defined by $\sigma : (s_n)_n \to (s_{n+1})_n$.

Lemma 1 *Let $\Psi : \mathcal{D}_\Psi (\subset \mathbb{R}^2) \to \mathbb{R}^2$ be a continuous map. Suppose there are two oriented rectangles $\widetilde{\mathcal{R}}_0$ and $\widetilde{\mathcal{R}}_1$ and four compact and pairwise disjoint sets $\mathcal{H}_{i,j} \subset \mathcal{R}_i \cap \mathcal{D}_\Psi$ such that*

$$(\mathcal{H}_{i,j}, \Psi) : \widetilde{\mathcal{R}}_i \Bumpeq\!\!\!\!\!\!\!\longrightarrow \widetilde{\mathcal{R}}_j, \ \forall\, i,j \in \{0,1\}.$$

Then, for each two-sided sequence $\boldsymbol{s} := (s_n) \in \Sigma_2$, there exists a sequence of points $(z_n)_{n\in\mathbb{Z}}$ in $\mathcal{D}_\Psi$ with $z_{n+1} = \Psi(z_n)$, $\forall\, n \in \mathbb{Z}$, such that

$$z_n \in \mathcal{H}_{s_n, s_{n+1}}, \quad \forall\, n \in \mathbb{Z}$$

and, moreover, we can choose $(z_n)_n$ as a k-periodic sequence if $\boldsymbol{s}$ is k-periodic. Furthermore, if Ψ is one-to-one, there exists a compact set $\Lambda \subset \mathcal{D}_\Psi \cap (\mathcal{R}_0 \cup \mathcal{R}_1)$ which is invariant for Ψ and such that $\Psi|_\Lambda$ is topologically semiconjugate to the Bernoulli shift on two symbols

$$\begin{tikzcd} \Lambda \arrow[r, "\Psi"] \arrow[d, "h"'] & \Lambda \arrow[d, "h"] \\ \Sigma_2 \arrow[r, "\sigma"'] & \Sigma_2 \end{tikzcd}$$

with the continuous surjection h (making the diagram commutative) such that $h^{-1}(\boldsymbol{s})$ contains a k-periodic point of Ψ for every k-periodic sequence $\boldsymbol{s} \in \Sigma_2$.

In the setting of Lemma 1 it is possible to derive further information about dynamical properties of the map Ψ. For instance, we know that the topological entropy of $\Psi|_\Lambda$ is at least $\log 2$. Figure 1 illustrates the directed graph associated to Lemma 1.

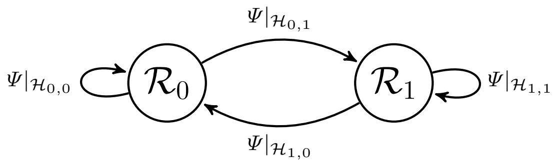

Fig. 1 Graph associated with the situation described in Lemma 1.

Lemma 1 can be extended to the case of a stretching with any crossing number $m \geq 2$. In this situation we can produce an invariant set Λ which is semiconjugate to the Bernoulli shift on m symbols in Σ_m. Actually, these results can be derived from a general criterion concerning an arbitrary sequence of oriented rectangles and maps (see [14, Theorem 2.2]) which we recall here for a later use in Section 4.

Lemma 2 *Assume there are double sequences of oriented rectangles $\widetilde{\mathcal{R}}_n$, compact sets $\mathcal{L}_n \subset \mathcal{R}_n$ and maps Ψ_n (for $n \in \mathbb{Z}$) such that*

$$(\mathcal{L}_n, \Psi_n) : \widetilde{\mathcal{R}}_n \Leftrightarrow\!\!\!\!\rightarrow \widetilde{\mathcal{R}}_{n+1}, \ \forall\, n \in \mathbb{Z}.$$

Then, the following conclusions hold:

- *There exists a two-sided sequence $(z_n)_{n\in\mathbb{Z}}$ such that $z_n \in \mathcal{L}_n$ and $\Psi_n(z_n) = z_{n+1}$ for all $n \in \mathbb{Z}$;*
- *If there are integers p, q with $p < q$ such that $\widetilde{\mathcal{R}}_p = \widetilde{\mathcal{R}}_q$, then there is a finite sequence $(z_n)_{p\leq n\leq q}$ with $z_n \in \mathcal{L}_n$ and $\Psi_n(z_n) = z_{n+1}$ for each $n = p, \ldots, q-1$, and such that $z_q = z_p$.*

In the special case when $\Psi_n = \Psi$ for all $n \in \mathbb{Z}$, the second instance of Lemma 2 guarantees the existence of a fixed point for Ψ^{q-p}, that is a periodic point of Ψ with period equal to $q - p$. By a suitable choice of the sets $\mathcal{L}_n$ it will be possible to prove that $q - p$ is the minimal period.

3 Main results

In this section we prove Theorem 1 as an application of Lemma 1 to the Poincaré map associated to system (3). Accordingly, as a first step, we make sure to have such Poincaré map globally defined on the plane. We extend G on the whole real line by setting:

$$\widetilde{G}(x) := \begin{cases} G(0) & \text{if } x < 0 \\ G(x) & \text{if } 0 \leq x \leq 1 \\ G(1) & \text{if } x > 1, \end{cases} \tag{4}$$

which is still differentiable in $\mathbb{R}$ with a locally Lipschitz continuous derivative $\widetilde{g} := \widetilde{G}'$. Thanks to (4) all the solutions of

$$\ddot{x} - w_\varepsilon(t)\widetilde{g}(x) = 0 \tag{5}$$

are defined for all $t \in \mathbb{R}$. Such a modification does not affect our study since, by a suitable form of the maximum principle, we can prove that solutions of (5) which are frequently in $]0, 1[$ must necessarily have range in $]0, 1[$. Indeed we prove the following:

Lemma 3 *Assume $x(t)$ is a solution of* (5) *and that there is a double sequence $(t_n)_{n\in\mathbb{N}}$ such that $t_n \to \pm\infty$ as $n \to \pm\infty$, $x(t_n) \le 1$ for all $n \in \mathbb{Z}$ and there is a $k \in \mathbb{Z}$ such that $x(t_k) < 1$. Then $x(t) < 1$ for all $t \in \mathbb{R}$. Similarly, if $x(t_n) \ge 0$ for all $n \in \mathbb{Z}$ and there is a $k \in \mathbb{Z}$ such that $x(t_k) > 0$, then $x(t) > 0$ for all $t \in \mathbb{R}$.*

Proof Without loss of generality we assume that t_n is strictly increasing. Suppose (by contradiction) there is $t_* \in \mathbb{R}$ such that $x(t_*) > 1$. We have $t_* \in]t_n, t_{n+1}[$ for some $n \in \mathbb{Z}$ and let $[t', t'']$ be the maximal interval such that $t_* \in [t', t''] \subset [t_n, t_{n+1}]$ and $x(t) > 1$ in $]t', t''[$. By the continuity of x and the unique solvability of Cauchy problems for (5) we have that $x(t') = x(t'') = 1$, $\dot{x}(t') > 0$ and $\dot{x}(t'') < 0$. Hence:

$$0 > \dot{x}(t'') - \dot{x}(t') = \int_{t'}^{t''} w_\varepsilon(t)\tilde{g}(x(t))dt = 0,$$

that is a contradiction. We have thus proved that $x(t) \le 1$ for all $t \in \mathbb{R}$. If $x(\hat{t}) = 1$ for some $\hat{t}$, then $\dot{x}(\hat{t}) = 0$ and by the unique solvability of the Cauchy problems for (5), we have $x(t) \equiv 1$. This cannot happen in our case because $x(t_k) < 1$. Hence $x(t) < 1$ for every t. A similar argument applies to prove that $x(t) > 0$ for every t. ☐

In what follows, in order to simplify the notation we will write G and g in place of $\widetilde{G}$ and $\widetilde{g}$.

On the time interval $[\tau, T]$ all the solutions of (1) are in fact solutions of the autonomous equation

$$\begin{cases} \dot{x} = y \\ \dot{y} = g(x) \end{cases} \tag{6}$$

therefore the pairs $(x(t), y(t)) = (x(t), \dot{x}(t))$ lie on the level lines of the energy

$$E(x, y) := \frac{y^2}{2} - G(x). \tag{7}$$

Lemma 4 *Let*

$$G_* := \min_{x\in[0,1]} G(x)$$

and let us fix any E_0 such that

$$E_0 \ge \frac{1}{2(T-\tau)^2} - G_*. \tag{8}$$

We have that every solution (x, y) of (6) *with energy $E(x(t), y(t)) = E_0$ and $x(\tau) \in [0, 1]$ satisfies $x(T) \ge 1$ if $y(\tau) > 0$ and $x(T) \le 0$ if $y(\tau) < 0$.*

Proof For all $t \in [\tau, T]$ we have that:

$$\dot{x}(t)^2 = y(t)^2 = 2(G(x(t)) + E_0) \ge 2(G_* + E_0) > 0,$$

by (8) and, thus, either $\dot{x}(t) > 0$ for all $t \in [\tau, T]$ or $\dot{x}(t) < 0$ for all $t \in [\tau, T]$. In both cases we can estimate:

$$|x(T) - x(\tau)| = \int_\tau^T |\dot{x}(t)|dt \geq (T - \tau)\sqrt{2(G_* + E_0)} \geq 1$$

again by assumption (8), and the thesis follows. $\square$

For each $a \in]0, a_0]$ and $b \in [b_0, 1[$ let us introduce the following "rectangular" domains

$$\mathcal{R}_0 := \{(x, y) : 0 \leq x \leq a \text{ and } y^2 \leq 2(G(x) + E_0)\}$$
$$\mathcal{R}_1 := \{(x, y) : b \leq x \leq 1 \text{ and } y^2 \leq 2(G(x) + E_0)\}$$

where E_0 satisfies (8) (see Figure 2). The sets $\mathcal{R}_0$ and $\mathcal{R}_1$ will be the supports of two corresponding oriented rectangles. Indeed we choose an orientation on $\mathcal{R}_0$ and $\mathcal{R}_1$ by setting:

$$\mathcal{R}_0^{\text{left}} := \{0\} \times \left[-\sqrt{2(G(0) + E_0)}, 0\right]$$
$$\mathcal{R}_0^{\text{right}} := \{a\} \times \left[-\sqrt{2(G(a) + E_0)}, \sqrt{2(G(a) + E_0)}\right]$$

and, symmetrically:

$$\mathcal{R}_1^{\text{left}} := \{1\} \times \left[0, \sqrt{2(G(1) + E_0)}\right]$$
$$\mathcal{R}_1^{\text{right}} := \{b\} \times \left[-\sqrt{2(G(b) + E_0)}, \sqrt{2(G(b) + E_0)}\right].$$

For later use we also define:

$$\mathcal{R}_0^{\text{top}} := \left\{\left(x, \sqrt{2(G(x) + E_0)}\right) : 0 \leq x \leq a\right\}$$
$$\mathcal{R}_0^{\text{bot}} := \left\{\left(x, -\sqrt{2(G(x) + E_0)}\right) : 0 \leq x \leq a\right\}$$
$$\mathcal{R}_1^{\text{top}} := \left\{\left(x, -\sqrt{2(G(x) + E_0)}\right) : b \leq x \leq 1\right\}$$
$$\mathcal{R}_1^{\text{bot}} := \left\{\left(x, \sqrt{2(G(x) + E_0)}\right) : b \leq x \leq 1\right\}.$$

Lemma 5 *Fix δ_1, δ_2 with $0 < \delta_1 < \delta_2 < a$. Then there is $\varepsilon_0 = \varepsilon_0(\delta_1, \delta_2, a) > 0$ such that the following properties hold for the solutions $(x(t), y(t))$ of* (3) *on the interval $[0, \tau]$ whenever $\varepsilon \in]0, \varepsilon_0[$:*

1. *if $(x(0), y(0)) \in \mathcal{R}_0$ and $x(0) = \delta_2$ and $[0, t_1]$ is the maximal interval in $[0, \tau]$ such that $(x(t), y(t)) \in \mathcal{R}_0 \cap \{(x, y) : x \geq \delta_1\}$ for all $t \in [0, t_1]$, then $t_1 < \tau$, $\delta_1 < x(t) < a$ for all $t \in [0, t_1]$ and $y(t_1) = \sqrt{2(G(x(t_1)) + E_0)}$;*
2. *if $(x(t_0), y(t_0)) \in \mathcal{R}_0$, $x(t_0) \leq \delta_2$, $y(t_0) \geq 0$ for some $t_0 \in [0, \tau]$ and $[t_0, t_1]$ is the maximal interval in $[t_0, \tau]$ such that $(x(t), y(t)) \in \mathcal{R}_0$ for all $t \in [t_0, t_1]$, then $x(t) < a$ for all $t \in [t_0, t_1]$.*

Proof We let

$$
\begin{aligned}
g_0 &:= \min_{s\in[\delta_1,a]} g(s)\\
M &:= \max_{(x,y)\in\mathcal{R}_0} |y| = \sqrt{2(G(a)+E_0)}\\
r &:= \min\left\{\tau, \frac{\delta_2-\delta_1}{M}, \frac{a-\delta_2}{M}\right\}\\
\varepsilon_0 &:= \frac{g_0}{2M} \min_{t_0\in[0,\tau-r]} \int_{t_0}^{t_0+r} A(t)dt
\end{aligned}
$$

and remark that $\varepsilon_0 > 0$ by our assumptions on g and A. In what follows we fix any $\varepsilon \in\,]0,\varepsilon_0[$.

In order to prove Statement 1 we point out that the solution $(x(t),y(t))$ may exit the set $\mathcal{R}_0 \cap \{x \geq \delta_1\}$ only through one of the vertical lines $\{\delta_1\}\times\,]-\infty,0[$ and $\{a\}\times[0,+\infty[$ or through $\mathcal{R}_0^{\rm top}$ and $\mathcal{R}_0^{\rm bot}$. If we use the energy $E(x,y)$ in (7) and, along solutions of (3) for $t\in[0,t_1]$, we compute:

$$
\dot{E}(x(t),y(t)) = -g(x(t))y(t) + y(t)\left[1+\frac{A(t)}{\varepsilon}\right]g(x(t)) = \frac{A(t)}{\varepsilon}y(t)g(x(t)),
$$

we immediately deduce that $\dot{E}(x(t),y(t)) \leq 0$ when $y(t)\leq 0$. Therefore, as long as $y(t)\leq 0$ for $t\in[0,t_1]$, $E(x(t),y(t))$ cannot increase above the value E_0 and the solution $(x(t),y(t))$ cannot exit $\mathcal{R}_0$ through $\mathcal{R}_0^{\rm bot}$. Thus, if we assume by contradiction that Statement 1 does not hold, then only the following three possibilities can occur: $t_1=\tau$ or $x(t_1)=\delta_1$ or $x(t_1)=a$. Since:

$$
|x(t_1)-\delta_2| = |x(t_1)-x(0)| = \left|\int_0^{t_1} y(t)dt\right| \leq Mt_1,
$$

in all three cases we have $t_1\geq r$. Hence, we obtain the following contradiction:

$$
2M \geq y(t_1)-y(0) = \int_0^{t_1}\left(1+\frac{A(t)}{\varepsilon}\right)g(x(t))dt \geq \frac{g_0}{\varepsilon}\int_0^r A(t)dt > 2M \quad (9)
$$

since $\varepsilon<\varepsilon_0$.

In a similar way we can show that, if $(x(t),y(t))$ is a solution as in Statement 2 which also satisfies $x(t_1)=a$, then again we have that $t_1-t_0\geq r$ and a contradiction like (9) is obtained. $\square$

A symmetric result holds in $\mathcal{R}_1$ and can be proved in a similar way.

Lemma 6 *Fix δ_1,δ_2 with $b<\delta_2<\delta_1<1$. Then there is $\varepsilon_1=\varepsilon_1(\delta_1,\delta_2,b)>0$ such that the following properties hold for the solutions $(x(t),y(t))$ of (3) on the interval $[0,\tau]$ whenever $\varepsilon\in\,]0,\varepsilon_1[$:*

1. *if $(x(0),y(0))\in\mathcal{R}_1$ and $x(0)=\delta_2$ and $[0,t_1]$ is the maximal interval in $[0,\tau]$ such that $(x(t),y(t))\in\mathcal{R}_1\cap\{(x,y): x\leq\delta_1\}$ for all $t\in[0,t_1]$, then $t_1<\tau$, $b<x(t)<\delta_1$ for all $t\in[0,t_1]$ and $y(t_1)=-\sqrt{2(G(x(t_1))+E_0)}$;*

2. *if* $(x(t_0), y(t_0)) \in \mathcal{R}_1$, $x(t_0) \geq \delta_2$, $y(t_0) \leq 0$ *for some* $t_0 \in [0,\tau]$ *and* $[t_0, t_1]$ *is the maximal interval in* $[t_0, \tau]$ *such that* $(x(t), y(t)) \in \mathcal{R}_1$ *for all* $t \in [t_0, t_1]$, *then* $x(t) > b$ *for all* $t \in [t_0, t_1]$.

Let $\Phi_s^t(z) = (x(t; s, z), y(t; s, z))$ be the solution $(x(t), y(t))$ of system (3) such that $(x(s), y(s)) = z$. For $i, j \in \{0,1\}$ we define the following compact sets:

$$\begin{aligned} \mathcal{H}_i &:= \{z \in \mathcal{R}_i : \Phi_0^t(z) \in \mathcal{R}_i \ \forall\, t \in [0,\tau]\}, \\ \mathcal{H}_{i,j} &:= \{z \in \mathcal{H}_i : \Phi_0^T(z) \in \mathcal{R}_j\}. \end{aligned} \tag{10}$$

Now we are in position to check the SAP property for the map $\Psi = \Phi_0^T$ with $\mathcal{D}_\Psi = \mathbb{R}^2$.

Lemma 7 *There exists* $\varepsilon^* = \varepsilon^*(a,b) > 0$ *such that for all* $\varepsilon \in \,]0, \varepsilon^*[$ *we have that* $(\mathcal{H}_{i,j}, \Phi_0^T) : \widetilde{\mathcal{R}}_i \Leftrightarrow\!\!\!\!\rightarrow \widetilde{\mathcal{R}}_j$ *for each* $i, j \in \{0,1\}$.

Proof Let $\varepsilon^* = \min\{\varepsilon_0, \varepsilon_1\}$, where ε_0 is given by Lemma 5 with the choices $\delta_1 = a/3$ and $\delta_2 = 2a/3$ and ε_1 is given by Lemma 6 with $\delta_1 = 1 - (1 - b)/3 = (2+b)/3$ and $\delta_2 = 1 - 2(1-b)/3 = (1+2b)/3$. We fix any $\varepsilon \in \,]0, \varepsilon^*[$ and show explicitly that $(\mathcal{H}_{0,j}, \Phi_0^T) : \widetilde{\mathcal{R}}_0 \Leftrightarrow\!\!\!\!\rightarrow \widetilde{\mathcal{R}}_j$ with $j \in \{0,1\}$. The other two situations are completely symmetric and therefore their proof is omitted. Figure 2 provides an illustration of a path γ crossing $\mathcal{R}_0$, which is stretched by Φ_0^T across $\mathcal{R}_0$ and $\mathcal{R}_1$.

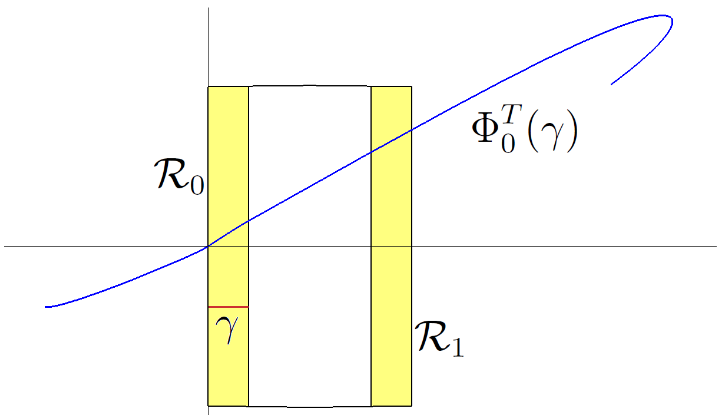

Fig. 2 Example of a path stretched by the Poincaré map. For the present example we have taken $g(x) = 2x(1-x)(\frac{1}{2} - x)$, $A(t) = \sin^+(2\pi t)$, so that $\tau = 1/2$ and $T = 1$. The sets $\mathcal{R}_0$ and $\mathcal{R}_1$ are defined for $a = 0.2$, $b = 0.8$ and $E_0 = y_0^2/2$ with $y_0 = 2.1$ so that (8) is satisfied. The figure shows the effect of the stretching of path γ through the Poincaré map Φ_0^T for $\varepsilon = 5 \times 10^{-3}$.

Now the analytical proof follows. Let $\gamma : [0,1] \to \mathcal{R}_0$ be a continuous curve such that $\gamma(0) \in \mathcal{R}_0^{\text{left}}$ and $\gamma(1) \in \mathcal{R}_0^{\text{right}}$. Since γ is continuous, we can define:

$$s_1 := \min\{s \in [0,1] : \gamma(s) \in \{\delta_2\} \times \mathbb{R}\}$$

and we have $0 < s_1 < 1$ and $\gamma(s) \in [0,\delta_2] \times \mathbb{R} \cap \mathcal{R}_0$ for all $s \in [0, s_1]$. By Lemma 5.1, $\Phi_0^t(\gamma(s_1))$ stays above $\mathcal{R}_0^{\mathrm{top}}$ for some $t \in]0,\tau[$, thus we can define:

$$s_2 := \inf \Big\{ s \in [0, s_1] : \exists t \in [0,\tau] \text{ s.t. either } x(t;0,\gamma(s)) > a \text{ or } \\ y(t;0,\gamma(s)) > \sqrt{2[G(x(t;0,\gamma(s))) + E_0]} \Big\}$$

and we have $0 < s_2 < s_1$. Moreover, by the continuous dependence on initial data, we deduce that $\Phi_0^t(\gamma(s_2)) \in \mathcal{R}_0$ for all $t \in [0,\tau]$ and, in particular, either $y(\tau;0,\gamma(s_2)) = \sqrt{2[G(x(\tau;0,\gamma(s_2))) + E_0]}$ or $x(\tau;0,\gamma(s_2)) = a$. Let us see that the second case cannot hold. Indeed, we surely have $y(\tau;0,\gamma(s_2)) > 0$ and, hence, we consider:

$$t_0 := \min\{t \in [0,\tau] : y(t;0,\gamma(s_2)) \geq 0\} < \tau.$$

so that $y(t;0,\gamma(s_2)) \geq 0$ for all $t \in [t_0,\tau]$. If $t_0 = 0$ then $x(t_0;0,\gamma(s_2)) \leq \delta_2$ by construction, while, if $t_0 > 0$, then $\dot{x}(t;0,\gamma(s_2)) = y(t;0,\gamma(s_2)) < 0$ for all $t \in [0,t_0[$ and again $x(t_0;0,\gamma(s_2)) \leq \delta_2$. Therefore, Lemma 5.2 applies and we deduce that $x(\tau;0,\gamma(s_2)) < a$ and, thus, that $y(\tau;0,\gamma(s_2)) = \sqrt{2[G(x(\tau;0,\gamma(s_2))) + E_0]}$. Hence, we can define

$$s_3 := \sup\{s \in [0,s_2] : \exists t \in [0,\tau] \text{ s.t. } x(t;0,\gamma(s)) = 0\}$$

and note that $0 \leq s_3 < s_2$ and $x(\tau;0,\gamma(s_3)) = 0$. By construction, we have that $\gamma([s_3,s_2]) \subset \mathcal{H}_0$ since $\Phi_0^t(\gamma(s)) \in \mathcal{R}_0$ for all $t \in [0,\tau]$ and all $s \in [s_3,s_2]$. Moreover $\Phi_0^\tau(\gamma(s_3)) \in \mathcal{R}_0^{\mathrm{left}}$ and $\Phi_0^\tau(\gamma(s_2)) \in \mathcal{R}_0^{\mathrm{top}}$.

Now, since the flow generated by (3) during the time interval $[\tau,T]$ coincides with the one generated by (6), the region $\{E(x,y) \leq E_0\}$, which contains both $\mathcal{R}_0$ and $\mathcal{R}_1$, is invariant for Φ_τ^t as t ranges in $[\tau,T]$. Therefore $\Phi_0^T(\gamma(s)) \in \{E(x,y) \leq E_0\}$ for all $s \in [s_3,s_2]$. Moreover:

$$\Phi_0^\tau(\gamma(s_3)) \in \mathcal{R}_0^{\mathrm{left}} \Longrightarrow \begin{cases} x(T;0,\gamma(s_3)) \leq 0 \\ y(T;0,\gamma(s_3)) \leq 0 \end{cases}$$

$$\Phi_0^\tau(\gamma(s_2)) \in \mathcal{R}_0^{\mathrm{top}} \Longrightarrow \begin{cases} x(T;0,\gamma(s_2)) \geq 1 \\ y(T;0,\gamma(s_2)) > 0 \end{cases}$$

by Lemma 4. We then define:

$$\begin{aligned} s_4 &:= \max\{s \in [s_3,s_2] : x(T;0,\gamma(s)) = 0\} \\ s_5 &:= \min\{s \in [s_4,s_2] : x(T;0,\gamma(s)) = a\} \\ s_6 &:= \max\{s \in [s_5,s_2] : x(T;0,\gamma(s)) = b\} \\ s_7 &:= \min\{s \in [s_6,s_2] : x(T;0,\gamma(s)) = 1\} \end{aligned}$$

and note that the above construction and the properties of the flow of (6) imply that $\Phi_0^T(\gamma(s)) \in \mathcal{R}_0$ for all $s \in [s_4,s_5]$ and $\Phi_0^T(\gamma(s)) \in \mathcal{R}_1$ for all $s \in [s_6,s_7]$, while $\Phi_0^T(\gamma(s_4)) \in \mathcal{R}_0^{\mathrm{left}}$, $\Phi_0^T(\gamma(s_5)) \in \mathcal{R}_0^{\mathrm{right}}$, $\Phi_0^T(\gamma(s_6)) \in \mathcal{R}_1^{\mathrm{right}}$ and $\Phi_0^T(\gamma(s_7)) \in \mathcal{R}_1^{\mathrm{left}}$. This shows that $(\mathcal{H}_{0,j},\Phi_0^T) : \widetilde{\mathcal{R}}_0 \Longleftrightarrow\!\!\!\!\!\rightarrow \widetilde{\mathcal{R}}_j$ for $j \in \{0,1\}$.

□

We are ready now to conclude the proof.

Proof (Proof of Theorem 1) Let the pair (a, b) be fixed as in the statement of the theorem, let $\varepsilon^* = \varepsilon^*(a, b) > 0$ be given by Lemma 7 and let us fix $\varepsilon \in]0, \varepsilon^*[$. Then we have that $(\mathcal{H}_{i,j}, \Phi_0^T) : \widetilde{\mathcal{R}}_i \Leftrightarrow\!\!\!\rightarrow \widetilde{\mathcal{R}}_j$ for each $i, j \in \{0, 1\}$ by Lemma 7. Therefore, Lemma 1, applied to $\Psi = \Phi_0^T$, grants that for any non-trivial two-sided sequence $\boldsymbol{s} = (s_n)_{n\in\mathbb{Z}} \in \{0, 1\}^{\mathbb{Z}}$ there exists a two-sided sequence $(z_n)_{n\in\mathbb{Z}}$ in the plane such that $z_{n+1} = \Phi_0^T(z_n)$ and

$$z_n \in \mathcal{H}_{s_n,s_{n+1}} \qquad \forall\, n \in \mathbb{Z},$$

with $(z_n)_n$ k-periodic whenever $\boldsymbol{s}$ is k-periodic. We will show that the function $u_{\boldsymbol{s},\varepsilon}(t) := x(t; 0, z_0)$ satisfies all the requirements in the statement of the theorem.

First of all, by the T-periodicity of w_ε, we have that

$$(u_{\boldsymbol{s},\varepsilon}(nT), \dot{u}_{\boldsymbol{s},\varepsilon}(nT)) = \Phi_0^{nT}(z_0) = (\Phi_0^T)^n(z_0) = z_n \in \mathcal{H}_{s_n,s_{n+1}} \subset \mathcal{H}_{s_n} \subset \mathcal{R}_{s_n}$$

for all $n \in \mathbb{Z}$. In particular we can apply Lemma 3 with the choice $t_n = nT$, $n \in \mathbb{Z}$: we have that $0 \le u_{\boldsymbol{s},\varepsilon}(t_n) \le 1$ for all $n \in \mathbb{Z}$ and, since the sequence $\boldsymbol{s}$ is nontrivial, there are $h, k \in \mathbb{Z}$, with $h \ne k$, such that $z_h \in \mathcal{R}_0$, $z_k \in \mathcal{R}_1$ and, thus, $u_{\boldsymbol{s},\varepsilon}(t_h) \le a < 1$ and $u_{\boldsymbol{s},\varepsilon}(t_k) \ge b > 0$. Therefore we can infer that $0 < u_{\boldsymbol{s},\varepsilon}(t) < 1$ for all $t \in \mathbb{R}$ and that $u_{\boldsymbol{s},\varepsilon}$ is in fact a solution of (1).

Moreover, if the sequence $\boldsymbol{s}$ is k-periodic, we have that $u_{\boldsymbol{s},\varepsilon}$ is kT-periodic since

$$(u_{\boldsymbol{s},\varepsilon}(0), \dot{u}_{\boldsymbol{s},\varepsilon}(0)) = z_0 = z_k = \Phi_0^{kT}(z_0) = (u_{\boldsymbol{s},\varepsilon}(kT), \dot{u}_{\boldsymbol{s},\varepsilon}(kT)).$$

Finally, Statement 2 also holds by the very definition of the set $\mathcal{H}_i$ (10) since $\Phi_0^{nT}(z_0) = z_n \in \mathcal{H}_{s_n}$. □

Remark 1 The same argument employed for the proof of Theorem 1 can be used to provide an extension of our result to a class of more general weight functions. Indeed, let us suppose that $A : \mathbb{R} \to \mathbb{R}$ is a T-periodic locally integrable function such that there are points

$$0 = \sigma_0 < \tau_0 < \sigma_1 < \tau_1 < \cdots < \sigma_{m-1} < \tau_{m-1} < \sigma_m = T$$

such that, for all $\ell = 0, \ldots, m-1$,

$(A1')$ $A(t) = 0$ for a. e. $t \in [\tau_\ell, \sigma_{\ell+1}]$ and $A(t) > 0$ for a. e. $t \in [\sigma_\ell, \tau_\ell]$.

In this case, if we choose

$$E_0 \ge \frac{1}{2\min_{0\le\ell\le m-1}(\sigma_{\ell+1} - \tau_\ell)^2} - G_*$$

(see (8)) we can construct the oriented rectangles $\widetilde{\mathcal{R}}_0$ and $\widetilde{\mathcal{R}}_1$ as above. For each interval $[\sigma_\ell, \tau_\ell]$ we can repeat the proof of Lemma 5 we gave for the interval $[0, \tau]$ and obtain a corresponding constant ε_0^ℓ. Similarly, we can reproduce

Lemma 6 on each interval $[\sigma_\ell, \tau_\ell]$ and obtain a corresponding constant ε_1^ℓ. Next, following (10), we introduce the sets

$$\mathcal{H}_i^\ell := \{z \in \mathcal{R}_i : \Phi_{\sigma_\ell}^t(z) \in \mathcal{R}_i \;\forall\, t \in [\sigma_\ell, \tau_\ell]\},$$
$$\mathcal{H}_{i,j}^\ell := \{z \in \mathcal{H}_i^\ell : \Phi_{\sigma_\ell}^{\sigma_{\ell+1}}(z) \in \mathcal{R}_j\}$$

and, arguing as in Lemma 7 we can prove the following

Lemma 8 *There exists $\varepsilon^* = \varepsilon^*(a,b) > 0$ such that for all $\varepsilon \in]0, \varepsilon^*[$ we have that $(\mathcal{H}_{i,j}^\ell, \Phi_{\sigma_\ell}^{\sigma_{\ell+1}}) : \widetilde{\mathcal{R}}_i \Leftrightarrow\!\!\!\rightarrow \widetilde{\mathcal{R}}_j$ for each $i,j \in \{0,1\}$ and each $\ell = 0, \ldots, m-1$.*

The constant ε^* is now defined as

$$\varepsilon^* := \min\{\varepsilon_0^\ell, \varepsilon_1^\ell : \ell = 0, \ldots, m-1\}.$$

Lemma 8 implies that the scheme of Figure 1 holds for each map $\Psi = \Phi_{\sigma_\ell}^{\sigma_{\ell+1}}$ with respect to the sets $\mathcal{H}_{i,j}^\ell$. Since the Poincaré map on one period is given by

$$\Phi_0^T = \Phi_{\sigma_{m-1}}^{\sigma_m} \circ \cdots \circ \Phi_{\sigma_0}^{\sigma_1},$$

we conclude that the same diagram of Figure 1 holds also for Φ_0^T, but each arrow corresponds to 2^{m-1} different itineraries. In other words, we have that for each $i,j \in \{0,1\}$, the Poincaré map Φ_0^T stretches $\widetilde{\mathcal{R}}_i$ to $\widetilde{\mathcal{R}}_j$ along the paths with crossing number 2^{m-1}. In this manner, under the assumptions $(G1)$ and $(A1')$, we have a version of Theorem 1 in which the obtained solutions realize a full dynamics on 2^m symbols.

4 More complicated dynamics

In this section we investigate the case in which solutions oscillate several times around $(1/2, 0)$ in the interval $[0, \tau]$. To achieve this result we need some further technical assumptions on the time map of an associated autonomous system which in turn allow us to compute the rotation number of the solutions. Accordingly, besides the basic hypotheses on $G(x)$ considered in Section 1, we suppose further that

$$(G2) \qquad G(0) = G(1) = 0 \quad \text{and} \quad G(x) > 0 \;\forall\, x \in]0,1[$$

and we still assume without loss of generality that the auxiliary position (4) holds. We denote by x^* a point in $]0,1[$ such that

$$G(x^*) = \max_{x \in [0,1]} G(x).$$

As a consequence of the above assumption, without loss of generality, we may assume that the constants a_0 and b_0 can be chosen so that

$$\min_{x \in [a_0, b_0]} G(x) = G(a_0) = G(b_0), \quad \text{with } a_0 < x^* < b_0\,.$$

Moreover, in order to make our analysis more transparent, we consider a T-periodic stepwise weight function $v_\mu(t)$ given by

$$(A2) \qquad v_\mu(t) = \begin{cases} \mu & \text{if } 0 \le t < \tau \\ 1 & \text{if } \tau \le t < T, \end{cases}$$

so that equation (1) reduces to

$$\ddot{x} - v_\mu(t) g(x) = 0. \tag{11}$$

As in the previous section, we perform a phase-plane analysis on the associated planar system

$$\begin{cases} \dot{x} = y \\ \dot{y} = v_\mu(t) g(x) \end{cases} \tag{12}$$

and we denote again by $\Phi_s^t(z)$ the value at time t of the solution of (12) such that $(x(s), y(s)) = z$.

When $t \in [0, \tau]$, (12) is the autonomous system

$$\begin{cases} \dot{x} = y \\ \dot{y} = \mu g(x) \end{cases} \tag{13}$$

whose solutions run on the level sets of the energy function

$$E_\mu(x, y) := \frac{y^2}{2} - \mu G(x)$$

and are periodic orbits contained in the strip $0 < x < 1$ if $-\mu G(a_0) \le E_\mu(x, y) < 0$, while the level set $E_\mu(x, y) = 0$ contains the two heteroclinic orbits connecting the saddle points $(0, 0)$ and $(1, 0)$. Indeed, for each $e \in [-\mu G(a_0), 0[$ there exist exactly two values x_0, x_1 such that $G(x_0) = G(x_1) = -e/\mu$ with $x_0 \in]0, a_0]$ and $x_1 \in [b_0, 1[$. Viceversa, for each $x_0 \in]0, a_0]$ (respectively, for each $x_1 \in [b_0, 1[$) there exists a unique periodic orbit of (13) passing through $(x_0, 0)$ (respectively, through $(x_1, 0)$) which crosses again the x-axis at only another point $(x_1, 0)$ with $b_0 \le x_1 < 1$ (respectively, $(x_0, 0)$ with $0 < x_0 \le a_0$) and whose period, denoted by $\mathcal{T}_\mu(x_0)$ is given by

$$\mathcal{T}_\mu(x_0) := \sqrt{\frac{2}{\mu}} p(x_0),$$

where

$$p(x_0) = p(x_1) := \int_{x_0}^{x_1} \frac{d\xi}{\sqrt{G(\xi) - G(x_0)}}.$$

In particular, all these periodic orbits turn around the point $P^* := (x^*, 0)$ in the clockwise sense. In the sequel it will be useful to introduce a polar coordinate systems with center at P^*, counting the angles in the clockwise sense starting from the half-line $\{(x^*, y) : y \le 0\}$. In this system, we denote by $\vartheta(t, Q)$ the angular coordinate associate with the solution $(x(t), y(t))$ of system (13) such that $(x(0), y(0)) = Q$, with $\vartheta(0, Q) \in]-\pi, \pi[$. Notice that with this position, $\vartheta(t, Q)$ is well defined for every initial point Q which does not belong to the vertical half-line $\{(x^*, y) : y \geq 0\}$.

On the time interval $[\tau, T]$, the solutions of (12) are again solutions of (6). In particular, Lemma 4 holds without any change.

Theorem 2 *Let G satisfy $(G1)$ and $(G2)$, v_μ be given as in $(A2)$, and a, b be fixed with $0 < a \leq a_0$ and $b_0 \leq a < 1$. For each $N \in \mathbb{N}$, $N \geq 1$, there exists $\mu_N^* = \mu_N^*(a, b) > 0$ such that for every $\mu > \mu_N^*$ and every sequence $\boldsymbol{s} = (\delta_n, k_n)_{n\in\mathbb{Z}} \in [\{0, 1\} \times \{0, 1, \ldots, 2N - 1\}]^{\mathbb{Z}}$ there exists at least a global solution $x_{\boldsymbol{s}}$ of (11) such that*

1. *$0 < x_{\boldsymbol{s}}(t) < 1$ for all $t \in \mathbb{R}$;*
2. *for all $n \in \mathbb{Z}$ one has that $x_{\boldsymbol{s}}(nT) < a$ if $\delta_n = 0$, while $x_{\boldsymbol{s}}(nT) > b$ if $\delta_n = 1$; moreover, $x_{\boldsymbol{s}} - x^*$ vanishes exactly k_n times in the interval $]nT, nT + \tau[$;*
3. *if the sequence $\boldsymbol{s}$ is m-periodic for some $m \in \mathbb{N}$, then $x_{\boldsymbol{s}}$ is mT-periodic.*

Proof According to Lemma 4 in the present situation we have $G_* = 0$ and we fix $E_0 \geq \frac{1}{2(T-\tau)^2}$. Given a, b with

$$0 < a \leq a_0, \quad b_0 \leq b < 1,$$

we choose $x_0 \in]0, a[$ and $x_1 \in]b, 1[$ such that $G(x_0) = G(x_1)$.
Next, we consider $\mu > \mu_1$, for

$$\mu_1 := \max\left\{\frac{G(a) + E_0}{G(a) - G(x_0)}, \frac{G(b) + E_0}{G(b) - G(x_1)}\right\}.$$

Now we introduce two regions (depending on the parameter $\mu > \mu_1$) as follows

$$\mathcal{S}_0 := \{(x, y) : 0 \leq x \leq a,\ 2\mu[G(x) - G(x_0)] \leq y^2 \leq 2(G(x) + E_0)\}$$
$$\mathcal{S}_1 := \{(x, y) : b \leq x \leq 1,\ 2\mu[G(x) - G(x_1)] \leq y^2 \leq 2(G(x) + E_0)\}.$$

By construction, $\mathcal{S}_i \subset \mathcal{R}_i$. Indeed, $\mathcal{S}_0$ is a rectangular domain bounded below and above by the level lines $y = \pm\sqrt{2(G(x) + E_0)}$, by the y-axis at the left and by the level line $y^2 = 2\mu[G(x) - G(x_0)]$ at the right. This is a consequence of the fact that the curves $y = \pm\sqrt{2(G(x) + E_0)}$ and $y^2 = 2\mu[G(x) - G(x_0)]$ cross exactly at one point in the strip $0 \leq x \leq a$ (actually the crossing point lies in $x_0 < x < a$) since G is strictly increasing on $[0, a_0]$ and $\mu > \mu_1$. Similarly, $\mathcal{S}_1$ is bounded at the left, above and below by the same level lines and at the right by the vertical line $x = 1$.

We choose a first orientation $(\mathcal{S}_i, \mathcal{S}_i^-)$ on $\mathcal{S}_i$, for $i = 0, 1$, by setting $\mathcal{S}_i^- := \mathcal{S}_i^{\text{left}} \cup \mathcal{S}_i^{\text{right}}$ with:

$$\mathcal{S}_0^{\text{left}} := \{0\} \times \left[-\sqrt{2(G(0) + E_0)}, 0\right]$$
$$\mathcal{S}_0^{\text{right}} := \{(x, y) : x_0 \leq x \leq a,\ 2\mu[G(x) - G(x_0)] = y^2 \leq 2(G(x) + E_0)\}$$

and, symmetrically:

$$\mathcal{S}_1^{\mathrm{left}} := \{1\} \times \left[0, \sqrt{2(G(1)+E_0)}\right]$$
$$\mathcal{S}_1^{\mathrm{right}} := \{(x,y) : b \le x \le x_1,\, 2\mu[G(x) - G(x_1)] = y^2 \le 2(G(x)+E_0)\}.$$

We also define:

$$\mathcal{S}_0^{\mathrm{top}} := \left\{(x,y) : 0 \le x \le a,\, y^2 \ge 2\mu[G(x) - G(x_0)],\, y = \sqrt{2(G(x)+E_0)}\right\}$$
$$\mathcal{S}_0^{\mathrm{bot}} := \left\{(x,y) : 0 \le x \le a,\, y^2 \ge 2\mu[G(x) - G(x_0)],\, y = -\sqrt{2(G(x)+E_0)}\right\}$$
$$\mathcal{S}_1^{\mathrm{top}} := \left\{(x,y) : b \le x \le 1,\, y^2 \ge 2\mu[G(x) - G(x_1)],\, y = -\sqrt{2(G(x)+E_0)}\right\}$$
$$\mathcal{S}_1^{\mathrm{bot}} := \left\{(x,y) : b \le x \le 1,\, y^2 \ge 2\mu[G(x) - G(x_1)],\, y = \sqrt{2(G(x)+E_0)}\right\}.$$

Indeed we will consider also the following (somehow complementary) orientation $(\mathcal{S}_i, \mathcal{S}_i^+)$ of $\mathcal{S}_i$, where:

$$\mathcal{S}_i^+ := \mathcal{S}_i^{\mathrm{bot}} \cup \mathcal{S}_i^{\mathrm{left}} \cup \mathcal{S}_i^{\mathrm{top}} \qquad i = 0, 1.$$

Observe that here $\mathcal{S}_i^{\mathrm{bot}} \cup \mathcal{S}_i^{\mathrm{left}}$ and $\mathcal{S}_i^{\mathrm{top}}$ are the two connected components of $\mathcal{S}^+$ that play the role of opposite sides of the topological rectangle $\mathcal{S}_i$.

We will show that the map Φ_0^τ stretches $(\mathcal{S}_i, \mathcal{S}_i^-)$ to $(\mathcal{S}_j, \mathcal{S}_j^+)$ multiple times, if μ is chosen large enough, while Φ_τ^T stretches $(\mathcal{S}_i, \mathcal{S}_i^+)$ to $(\mathcal{S}_j, \mathcal{S}_j^-)$, for each $\mu > \mu_1$. In order to do this, we now define the compact sets in $\mathcal{S}_i$ with respect to which the stretching along the paths occurs. Namely, for each $i, j \in \{0,1\}$ and $k \in \mathbb{N}$ we set:

$$\mathcal{H}_{i,j} := \{Q \in \mathcal{S}_i : \Phi_0^\tau(Q) \in \mathcal{S}_j\},$$
$$\mathcal{H}_{i,j}^k := \left\{Q \in \mathcal{H}_{ij} : \frac{\vartheta(\tau, Q)}{\pi} \in\,]i + |j-i| + 2k, i + |j-i| + 2k + 1[\right\},$$
$$\mathcal{K}_{i,j} := \{Q \in \mathcal{S}_i : \Phi_\tau^T(Q) \in \mathcal{S}_j\}.$$

Observe that a solution of (12) starting at time $t = 0$ from $Q \in \mathcal{H}_{ij}^k$ will cross the line $x = x^*$ exactly $|i - j| + 2k$ times before reaching the rectangle $\mathcal{S}_j$ at time $t = \tau$.

Claim For any $N \in \mathbb{N}$ and any μ such that:

$$\mu > \mu_N^* := \max\left\{\mu_1, 2\left[N\frac{p(x_0)}{\tau}\right]^2\right\} \tag{14}$$

we have that $\Phi_0^\tau : (\mathcal{S}_i, \mathcal{S}_i^-) \Leftrightarrow\!\!\!\!\!\!\!\!\longrightarrow^N (\mathcal{S}_j, \mathcal{S}_j^+)$, for each $i, j \in \{0,1\}$, with respect to the compact sets $\mathcal{H}_{ij}^k$, for $k = 1, \ldots, N$.

Indeed, let us consider the case $i = 0$ and let $\gamma : [0,1] \to \mathcal{S}_0$ be any path such that $\gamma(0) \in \mathcal{S}_0^{\mathrm{left}}$ and $\gamma(1) \in \mathcal{S}_0^{\mathrm{right}}$. We have that $\Phi_0^t(\gamma(0)) \in \,]-\infty, 0[\, \times \,]-\infty, 0[$ for all $t > 0$. On the other hand, $\Phi_0^t(\gamma(1))$ belongs to the level line

$$E_\mu(x,y) = -\mu G(x_0),$$

which is a periodic orbit of system (13) of period $\mathcal{T}_\mu(x_0)$. By (14) we have that $\tau > N\mathcal{T}_\mu(x_0)$. Passing to the polar coordinates, this in turn implies that $\vartheta(\tau,\gamma(1)) > 2N\pi$, while $\vartheta(\tau,\gamma(0)) < \pi/2$, therefore the interval spanned by the angle $\vartheta(\tau,\gamma(s))$ as s ranges in $[0,1]$ contains the interval $[\pi/2, 2N\pi]$.

We can now split the interval $[0,1]$ into some subintervals of the form $[s'_\ell, s''_\ell]$, as follows:

$$\begin{aligned}
s'_0 &:= \max\{s \in [0,1] : x(\tau; 0, \gamma(s)) \le 0\},\\
s''_0 &:= \min\{s \in \,]s'_0, 1] : \Phi_0^\tau(\gamma(s)) \in \mathcal{S}_0^{\mathrm{top}}\},\\
s'_1 &:= \max\left\{s \in \,]s''_0, 1] : 1 < \frac{\vartheta(\tau,\gamma(s))}{\pi} < 2, \Phi_0^\tau(\gamma(s)) \in \mathcal{S}_1^{\mathrm{bot}}\right\},\\
s''_1 &:= \min\{s \in \,]s'_1, 1] : \Phi_0^\tau(\gamma(s)) \in \mathcal{S}_1^{\mathrm{top}}\},
\end{aligned}$$

and recursively for $k = 1, \ldots, N-1$:

$$\begin{aligned}
s'_{2k} &:= \max\left\{s \in \,\big]s''_{2k-1}, 1\big] : 2k < \frac{\vartheta(\tau,\gamma(s))}{\pi} < 2k+1, \Phi_0^\tau(\gamma(s)) \in \mathcal{S}_0^{\mathrm{bot}}\right\},\\
s''_{2k} &:= \min\{s \in \,]s'_{2k}, 1] : \Phi_0^\tau(\gamma(s)) \in \mathcal{S}_0^{\mathrm{top}}\},\\
s'_{2k+1} &:= \max\left\{s \in \,]s''_{2k}, 1] : 2k+1 < \frac{\vartheta(\tau,\gamma(s))}{\pi} < 2k+2, \Phi_0^\tau(\gamma(s)) \in \mathcal{S}_1^{\mathrm{bot}}\right\},\\
s''_{2k+1} &:= \min\{s \in \,\big]s'_{2k+1}, 1\big] : \Phi_0^\tau(\gamma(s)) \in \mathcal{S}_1^{\mathrm{top}}\}.
\end{aligned}$$

For each $\ell = 0, 1, \ldots, 2N-1$, by the choice of the points s'_ℓ and s''_ℓ it follows that $\Phi_0^\tau(\gamma([s'_\ell, s''_\ell])) \in \mathcal{S}_j$, $\Phi_0^\tau(\gamma(s'_\ell)) \in \mathcal{S}_j^{\mathrm{left}} \cup \mathcal{S}_j^{\mathrm{bot}}$, $\Phi_0^\tau(\gamma(s''_\ell)) \in \mathcal{S}_j^{\mathrm{top}}$, and $\gamma([s'_\ell, s''_\ell]) \subset \mathcal{H}_{0,j}^k$, where $j \equiv \ell \mod 2$ and $k = \lfloor \ell/2 \rfloor$. We just remark here the small difference between the cases $\ell = 0$ and $1 \le \ell \le 2N-1$ that led us to the definition of $\mathcal{S}_i^+$: we actually have that $\Phi_0^\tau(\gamma(s'_0)) \in \mathcal{S}_0^{\mathrm{left}}$, while $\Phi_0^\tau(\gamma(s'_\ell)) \in \mathcal{S}_j^{\mathrm{bot}}$ for $1 \le \ell \le 2N-1$ and $j \equiv \ell \mod 2$. However, we now have shown that

$$(\mathcal{H}_{0,j}^k, \Phi_0^\tau) : (\mathcal{S}_0, \mathcal{S}_0^-) \Leftrightarrow\!\!\!\rightarrow (\mathcal{S}_j, \mathcal{S}_j^+) \qquad \text{for } j = 0, 1 \tag{15}$$

for each $k = 0, \ldots, N-1$. The remaining cases with $i = 1$ can be proved in a similar way.

Claim For any $\mu > \mu_1$ and any $i, j \in \{0,1\}$ we have that

$$(\mathcal{K}_{ij}, \Phi_\tau^T) : (\mathcal{S}_i, \mathcal{S}_i^+) \Leftrightarrow\!\!\!\rightarrow (\mathcal{S}_j, \mathcal{S}_j^-). \tag{16}$$

Again we will show the details only for the cases with $i = 0$ and leave the ones with $i = 1$ to the reader. Let $\gamma : [0,1] \to \mathcal{S}_0$ be any path such that $\gamma(0) \in \mathcal{S}_0^{\mathrm{left}} \cup \mathcal{S}_0^{\mathrm{bot}}$ and $\gamma(1) \in \mathcal{S}_0^{\mathrm{top}}$. Still Φ_τ^T maps the point $\gamma(0)$ in the third quadrant: if $\gamma(0) \in \mathcal{S}_0^{\mathrm{left}}$, it is just a consequence of the behavior of (12), while, if $\gamma(0) \in \mathcal{S}_0^{\mathrm{bot}}$, it comes from the choice of E_0 and Lemma 4. On the other hand, $\Phi_\tau^T(\gamma(1))$ lies in the half plane $x > 1$ again by the choice of E_0 and Lemma 4. Moreover the strip given by $\{(x,y) : E_1(x,y) \le E_0\}$ is invariant for Φ_τ^t for any $t \in [\tau, T]$. Hence we can define:

$$
\begin{aligned}
s_0' &:= \max\{s \in [0,1] : x(T;\tau,\gamma(s)) \le 0\},\\
s_0'' &:= \min\{s \in [s_0',1] : \Phi_\tau^T(\gamma(s)) \in \mathcal{S}_0^{\mathrm{right}}\},\\
s_1' &:= \max\{s \in [s_0'',1] : \Phi_\tau^T(\gamma(s)) \in \mathcal{S}_1^{\mathrm{right}}\},\\
s_1'' &:= \min\{s \in [s_1',1] : x(T;\tau,\gamma(s)) \ge 1\}.
\end{aligned}
$$

These choices imply that, for $j = 0,1$, $\Phi_\tau^T(\gamma([s_j', s_j''])) \subset \mathcal{S}_j$, $\Phi_\tau^T(\gamma(s_0')) \in \mathcal{S}_0^{\mathrm{left}}$, $\Phi_\tau^T(\gamma(s_0'')) \in \mathcal{S}_0^{\mathrm{right}}$, $\Phi_\tau^T(\gamma(s_1')) \in (S)_1^{\mathrm{right}}$, $\Phi_\tau^T(\gamma(s_1'')) \in \mathcal{S}_1^{\mathrm{left}}$ and $\gamma([s_j', s_j'']) \subset \mathcal{K}_{0,j}$. Thus, the claim is proved for $i = 0$.

As a consequence of Claim 1 and Claim 2, we have that the map $\Phi_0^T = \Phi_\tau^T \circ \Phi_0^\tau$ satisfy a SAP property of the form $(\mathcal{S}_i, \mathcal{S}_i^-) \Leftrightarrow\!\!\!\rightarrow (\mathcal{S}_j, \mathcal{S}_j^-)$, through the composition

$$(\mathcal{S}_i, \mathcal{S}_i^-) \Leftrightarrow\!\!\!\rightarrow (\mathcal{S}_h, \mathcal{S}_h^+) \Leftrightarrow\!\!\!\rightarrow (\mathcal{S}_j, \mathcal{S}_j^-),$$

where $i, h, j \in \{0,1\}$ can be chosen arbitrarily (the fact that the SAP property is preserved by the composition of maps easily follows from the definition [16]). To make the formula more precise, we should determine the compact subsets of $\mathcal{S}_i$ which are involved in the definition. Actually, from (15) and (16), we have that

$$(\mathcal{H}_{i,h}^k \cap \Phi_\tau^0(\mathcal{K}_{h,j}), \Phi_0^T) : (\mathcal{S}_i, \mathcal{S}_i^-) \Leftrightarrow\!\!\!\rightarrow (\mathcal{S}_j, \mathcal{S}_j^-), \tag{17}$$

where we recall that $\Phi_\tau^0 = (\Phi_0^\tau)^{-1}$.

Suppose now that $\mu > \mu_N^*$ is fixed and let $\boldsymbol{s} = (\delta_n, k_n)_{n\in\mathbb{Z}} \in [\{0,1\} \times \{0,1,\ldots,2N-1\}]^{\mathbb{Z}}$ be an arbitrary two-sided sequence. We show how to enter in the setting of Lemma 2, via the following positions.

For each $n \in \mathbb{Z}$ we take as oriented rectangle

$$\widetilde{\mathcal{R}}_n := \widetilde{\mathcal{S}}_{\delta_n}$$

and a constant sequence of maps

$$\Psi_n := \Phi_0^T = \Phi_{nT}^{nT+T}.$$

For the compact sets $\mathcal{L}_n$ we make the following observation. A solution with initial point in $\mathcal{S}_{\delta_n}$ (at the time nT), after the time τ will be in the same rectangle or in the other one according to the fact that k_n is even or odd, respectively. On the other hand, the index δ_{n+1} specifies in which rectangle the solution should be at the time $nT + T$. Therefore, in view of formula (17), we have to take, at any step n, $i = \delta_n$, $j = \delta_{n+1}$ and the intermediate index h will be determined according to the parity of k_n. Accordingly, we define

$$\mathcal{L}_n := \mathcal{H}^k_{\delta_n,h} \cap \Phi^0_\tau(\mathcal{K}_{h,\delta_{n+1}}), \tag{18}$$

with

$$k := \left\lfloor \frac{k_n}{2} \right\rfloor, \quad \text{and} \quad h \equiv k_n + \delta_n \mod 2.$$

Now we are in position to apply Lemma 2 to the sequence

$$(\mathcal{L}_n, \Psi_n) : \widetilde{\mathcal{R}}_n \Leftrightarrow\!\!\!\rightarrow \widetilde{\mathcal{R}}_{n+1}, \quad \forall\, n \in \mathbb{Z}.$$

In particular, given any sequence $(z_n)_{n\in\mathbb{Z}}$ with $z_{n+1} = \Phi^T_0(z_n)$ with $z_n \in \mathcal{L}_n$ for each $n \in \mathbb{Z}$, we have that the solution $(x(t), y(t)) = (x(t; 0, z_0), y(t; 0, z_0))$ of (12) satisfies the following properties:

1. $x(t)$ is a solution of (11) with $0 < x(t) < 1$ for all $t \in \mathbb{R}$ (this follows by the construction of the rectangular sets and also by Lemma 3).
2. $x(nT) < a_0$ if $\delta_n = 0$ and $x(nT) > b_0$ if $\delta_n = 1$. Moreover, $x(t) - x^*$ has exactly k_n simple zeros in the interval $]nT, nT + \tau[$ (this follows from the choice of the sets $\mathcal{H}^k_{i,h}$ and the definition (18)).
3. If $z_m = \Phi^{mT}_0(z_0) = z_0$ for some $m \geq 1$, then the corresponding solution $x(t)$ is mT-periodic (this situation occurs when the sequence $(\mathcal{L}_n, \widetilde{\mathcal{R}}_n)_n$ is m-periodic and this, in turn, follows whenever the sequence of symbols $\boldsymbol{s}$ is m-periodic).

Concerning the third property, observe that, if we choose the sequence $\mathcal{L}_n$ as a periodic sequence of minimal period m, then the corresponding mT-periodic solution $x(t)$ has mT as its minimal period.
In this manner all the assertions in the statement of Theorem 2 have been verified. □

Remark 2 Theorem 2 is stable with respect to small perturbations of the weight function $v_\mu(t)$. Indeed, it is possible to check that Claim 1 and Claim 2 are still valid if we perturb the right hand member of equation (12) by a sufficiently small term. More precisely, as a consequence of the theorem of continuous dependence of solutions [10, Lemma 3.2 and p. 28], we see that once we have fixed N and $\mu > \mu^*_N$, then the same conclusion of Theorem 2 holds for equation

$$x'' + cx' + w(t)g(x) = 0,$$

provided that $|c| < \delta$ and $\int_0^T |w(t) - v_\mu(t)|dt < \delta$, where $\delta = \delta_{N,\mu} > 0$ is a sufficiently small constant.

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